数理解析研究所講究録
第 1962 巻 2015 年 50-58

END-POINT MAXIMAL L^1 REGULARITY FOR A CAUCHY PROBLEM TO PARABOLIC EQUATIONS

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1. Introduction

In this summary, we consider maximal L^1-regularity of the Cauchy problem for parabolic equations in the non-reflexive homogeneous Besov space.

Let X be a Banach space and A be a closed linear operator in X with a densely defined domain $\mathcal{D}(A)$. Given $f \in L^\rho(0,T;X)$ $(1<\rho<\infty)$, we consider the abstract Cauchy problem with $0<t<T\le\infty$:

$$\begin{cases} \dfrac{d}{dt}u + Au = f, & t>0, \\ \qquad u(0)=0, & t=0 \end{cases} \tag{1.1}$$

Then it is called that A has maximal L^ρ regularity if there exists a unique solution $u \in W^{1,\rho}(0,T;X)\cap L^\rho(0,T;\mathcal{D}(A))$ to the abstract parabolic equation (1.1) and satisfies the estimate

$$\left\|\frac{d}{dt}u\right\|_{L^\rho(0,T;X)} + \|Au\|_{L^\rho(0,T;X)} \le C\|f\|_{L^\rho(0,T;X)}, \tag{1.2}$$

where C is a positive constant independent of f. In a general theory, maximal regularity is well established for any Banach space X that satisfies "Unconditional Martingale Difference" (called as UMD). See for the details [2], [4], [8], [13], [14], [15], [20], [21], [26]. On the other hand, maximal regularity on non-UMD Banach spaces, for instance non-reflexive Banach space such as L^1 or L^∞-like spaces, requires a different way to show it. When we consider the Cauchy problem for the linear parabolic equation the estimate for maximal regularity (1.2) reflects directly full regularity of the solution. Let u solve the Cauchy problem

$$\begin{cases} \partial_t u - \mathcal{L}_2 u = f, & t>0,\ x\in\mathbb{R}^n, \\ u(0,x)=u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.3}$$

where the operator $\mathcal{L}_2$ denotes the uniformly elliptic operator of second order, ∂_t denotes the partial derivative by t and u_0 and f are given initial and external data. Then general theory is stated avoding the end point spaces such as L^1 or L^∞ in both space and time variables. In the case of $\mathcal{L}_2=\Delta$, we explicitly proved maximal regularity on the homogenous Banach spaces [22], [23]. To state the result precisely, we first recall the definition

AMS Subject Classification: primary 35Q05, secondary 35L60, 75C05.

of the Besov space. Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity satisfying that

$$\sum_{j\in\mathbb{Z}} \widehat{\phi}_j(\xi) = 1$$

for all $\xi \neq 0$, where $\hat{\phi}$ is the Fourier transform of ϕ and $\operatorname{supp} \hat{\phi}_j \subset \{\xi \in \mathbb{R}^n | \ 2^{j-1} < |\xi| < 2^{j+1}\}$. For $s \in \mathbb{R}$ and $1 \le p, \sigma \le \infty$, we define the homogeneous Besov space $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ by

$$\dot{B}^s_{p,\sigma}(\mathbb{R}^n) = \{f \in \mathcal{S}^*/\mathcal{P}; \|f\|_{\dot{B}^s_{p,\sigma}} < \infty\}$$

with the norm

$$\|f\|_{\dot{B}^s_{p,\sigma}} \equiv \begin{cases} \Big(\sum_{j\in\mathbb{Z}} 2^{js\sigma}\|\phi_j * f\|_p^\sigma\Big)^{1/\sigma}, 1 \le \sigma < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\phi_j * f\|_p, \ \sigma = \infty \end{cases}$$

and $\mathcal{P}$ denotes all polynomials. We also introduce the inhomogeneous Besov spaces $B^s_{p,\sigma}(\mathbb{R}^n)$ by

$$B^s_{p,\sigma}(\mathbb{R}^n) = \{f \in \mathcal{S}^*; \|f\|_{B^s_{p,\sigma}} < \infty\}$$

with the norm

$$\|f\|_{B^s_{p,\sigma}} \equiv \begin{cases} \Big(\|\psi * f\|_p^\sigma + \sum_{j\ge 0} 2^{js\sigma}\|\phi_j * f\|_p^\sigma\Big)^{1/\sigma}, 1 \le \sigma < \infty, \\ \|\psi * f\|_p + \sup_{j\ge 0} 2^{js}\|\phi_j * f\|_p, \ \sigma = \infty \end{cases}$$

where ψ is a smooth cut off function with

$$\psi(\xi) + \sum_{j\ge 0} \hat{\phi}(\xi) \equiv 1$$

for all $\xi \in \mathbb{R}^n$ (cf. [5], [6], [25]).

One of a general result in the Besov spaces can be seen in [23]:

Proposition 1.1 (endpoint maximal regularity). *Let $\mathcal{L}_2 = \Delta$, $1 < \rho, \sigma \le \infty$ and $I = [0, T)$ be an interval with $T \le \infty$. For $f \in L^\rho(I; \dot{B}^0_{1,\rho}(\mathbb{R}^n))$ and $u_0 \in \dot{B}^{2(1-1/\rho)}_{1,\rho}(\mathbb{R}^n)$, let u be a solution of the Cauchy problem of the heat equation* (1.3). *Then there exists a constant $C_M > 0$ such that*

$$\|\partial_t u\|_{L^\rho(I;\dot{B}^0_{1,\rho})} + \|\nabla^2 u\|_{L^\rho(I;\dot{B}^0_{1,\rho})} \le C_M\Big(\|u_0\|_{\dot{B}^{2(1-1/\rho)}_{1,\rho}} + \|f\|_{L^\rho(I;\dot{B}^0_{1,\rho})}\Big).$$

Proposition 1.1 does not cover the end-point case $\rho = 1$, partially because the argument in the proof in [23] involves a duality structure and it is not clear if maximal L^1-regularity holds by applying the method utilized there. On the other hand, Danchin [10], [11] (see also Haspot [17]) obtained maximal regularity in the homogeneous Besov space for the case $\rho = 1$. In this paper, we reconsider maximal L^1-regularity in the Besov space and its optimality in the homogeneous Besov spaces.

2. Results for a constant coefficient case

Our main statement for the Cauchy problem for the heat equation (1.3) is the following:

Theorem 2.1 (optimal maximal L^1 regularity). *Let $\mathcal{L}_2 = \Delta$, $1 \le p \le \infty$. For $f \in L^1(\mathbb{R}_+; \dot{B}^0_{p,1}(\mathbb{R}^n))$ and $u_0 \in \dot{B}^0_{p,1}(\mathbb{R}^n)$ there exists a unique solution u to* (1.3) *which satisfies the estimate: There exists a positive constant $C_M > 0$ only depending on n, p such that*

$$\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})} + \left\|\nabla^2 u\right\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})} \le C_M \left(\|u_0\|_{\dot{B}^0_{p,1}} + \|f\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})}\right). \tag{2.1}$$

Besides if $f \equiv 0$, then the regularity condition for the initial data is optimal. Namely there exists a constant $C_m = C_m(n,p) > 0$ such that for all $u_0 \in \dot{B}^0_{p,1}(\mathbb{R}^n)$

$$C_m\|u_0\|_{\dot{B}^0_{p,1}} \le \|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})} + \left\|\nabla^2 u\right\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})}. \tag{2.2}$$

The upper estimate of (2.1) was obtained by Danchin [9], [10], [11] and Haspot [17] with $1 < p < \infty$ (see also Danchin-Mucha [12]). However our method to obtaining the estimates (2.1) seems very different from those existing arguments. In fact, our method admits the fractional order ellipic operator such as $\mathcal{L}_\alpha = (-\Delta)^{\alpha/2}$ for $\alpha > 0$ and an analogous estimate in Theorem 2.1 also holds. We state this version precisely in below (Theorem 2.9).

If we replace $u_0 \in \dot{B}^0_{p,1}(\mathbb{R}^n)$ into $u_0 \in \dot{B}^0_{p,\sigma}(\mathbb{R}^n)$ or $\dot{F}^0_{p,\sigma}(\mathbb{R}^n)$ for $1 < \sigma \le \infty$, then maximal regularity in $L^1(\mathbb{R}_+; \dot{B}^0_{p,\sigma}(\mathbb{R}^n))$ or $L^1(\mathbb{R}_+; \dot{F}^0_{p,\sigma}(\mathbb{R}^n))$ fails since the lower bound by the initial data and the strict inclusion result for the sub-sufix σ such as $\dot{B}^0_{p,1}(\mathbb{R}^n) \subsetneq \dot{B}^0_{p,\sigma}(\mathbb{R}^n)$. In particular the estimate in $L^1(\mathbb{R}_+; L^p(\mathbb{R}^n))$;

$$\int_0^\infty \|\Delta e^{t\Delta} u_0\|_p dt \le C\|u_0\|_p \tag{2.3}$$

generally fails. If $1 < p \le 2$, then $\dot{B}^0_{p,1} \subsetneq L^p = \dot{F}^0_{p,2} \subset \dot{B}^0_{p,2}$, and if $2 \le p < \infty$ then $\dot{B}^0_{p,1} \subsetneq \dot{B}^0_{p,2} \subset \dot{F}^0_{p,2} = L^p$ so that the estimate (2.3) contradicts the result (2.2) for general data u_0. The equivalence between the homogeneous Besov norm and the expression of the heat kernel is also pointed out in Bahouri-Chemin-Danchin [3] by the following form:

$$\int_0^\infty \|e^{t\Delta} u_0\|_p dt \simeq \|u_0\|_{\dot{B}^{-2}_{p,1}}.$$

See for the application of this expression to the initial boundary value problem for the incomressible Navier-Stokes equation, Cannone-Planchon-Schonbek [7].

Giga-Saal [16], proved maximal L^1-regularity over the class of Fourier transformed finite Radon measures $\mathcal{FM}(\mathbb{R}^n)$. Let $\mathcal{M}(\mathbb{R}^n)$ be a class of signed finite Radon measures and let

$$\mathcal{FM}(\mathbb{R}^n) \equiv \{f = \hat{\mu}, \mu \in \mathcal{M}(\mathbb{R}^n)\}$$

with the norm $\|f\|_{F\mathcal{M}} \equiv \|\mu\|_{\mathcal{M}}$, where $\|\mu\|_{\mathcal{M}}$ denotes the total variation of $\mu \in \mathcal{M}(\mathbb{R}^n)$.

Proposition 2.2 (Giga-Saal). *Let u be a solution to the Cauchy problem of the heat equation* (1.3) *with $\mathcal{L}_2 = \Delta$. Then there exists a constant $C > 0$ such that Then for*

$u_0 \in \mathcal{FM}(\mathbb{R}^n)$ *and* $f \in L^1(\mathbb{R}_+; \mathcal{FM}(\mathbb{R}^n))$ *maximal* L^1*-regularity for the heat equation holds:*

$$\|\partial_t u\|_{L^1(I;\mathcal{FM})} + \|\nabla^2 u\|_{L^1(I;\mathcal{FM})} \le C_M(\|u_0\|_{\mathcal{FM}} + \|f\|_{L^1(I;\mathcal{FM})}). \tag{2.4}$$

They applied this estimate for solving the Cauchy problem of the incompressible Navier-Stokes equations with the Coriolis force. Our result is a version of improvement of the Giga-Saal estimate (2.4) since the following embedding holds.

$$F\mathcal{M}(\mathbb{R}^n)/\{constant\} \hookrightarrow \dot{B}^0_{\infty,1}(\mathbb{R}^n).$$

In particular the embedding is continuous. For the case initial data is constant, then maximal regularity is trivial. If $f = 1$ and $u_0 = 0$ then $u(t,x) = t$ is a unique solution and again maximal regularity holds in $F\mathcal{M}$. The homogeneous Besov space can not include this case however the estimate itself is trivial.

As a corollary of Theorem 2.1, we obtain the lower estimate for $f \neq 0$ case.

Corollary 2.3. *Let* $\mathcal{L}_2 = \Delta$, $1 \le p \le \infty$ *and the constants* C_M *and* C_m *represents the upper bound of* (2.1) *and the lower bound of* (2.2), *respectively. If* $u_0 \in \dot{B}^0_{p,1}$ *and* $f \in L^1(\mathbb{R}_+; \dot{B}^0_{p,1})$ *satisfy*

$$C_M\|f\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})} < C_m\|u_0\|_{\dot{B}^0_{p,1}}$$

or

$$C_M\|u_0\|_{\dot{B}^0_{p,1}} < \|f\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})},$$

then there exists a constant $C(n,p) > 0$ *such that the solution to the heat equation* (1.3) *satisfies*

$$C\big(\|u_0\|_{\dot{B}^0_{p,1}} + \|f\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})}\big) \le \|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})} + \|\nabla^2 u\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})}.$$

For the case $u_0 = 0$, the lower estimate holds for the sum of the norm for $\partial_t u$ and $\nabla^2 u$ as

$$\|f\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})} \le \|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})} + \|\nabla^2 u\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})}.$$

On the other hand, for the case that $f = 0$, the lower estimate (2.2) holds for the each term of the right-hand side as

$$C^{-1}\|u_0\|_{\dot{B}^0_{p,1}(\mathbb{R}^n)} \le \|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1}(\mathbb{R}^n))},$$
$$C^{-1}\|u_0\|_{\dot{B}^0_{p,1}(\mathbb{R}^n)} \le \|\nabla^2 u\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1}(\mathbb{R}^n))},$$

which are derived from the following proposition.

Proposition 2.4. *For* $1 \le p \le \infty$, *let* $u_0 \in \dot{B}^0_{p,1}$.

(1) *Then there exists a constant* $C > 0$ *such that for any* $k \in \mathbb{Z}$ *it holds*

$$C^{-1}\sum_{\ell \le k}\|\phi_\ell * u_0\|_p \le \sum_{\ell \le k}\int_{2^{-2\ell}}^{2^{-2\ell+2}}\|\Delta e^{s\Delta}u_0\|_p ds \le C\sum_{j\in\mathbb{Z}}\min(1, e^{-2^{(j-k)}})\|\phi_j * u_0\|_p. \tag{2.5}$$

(2) *For $I = [0, T]$, there exists an integer $\tilde{\ell} = \left[-\frac{\log T}{2\log 2}\right]$ and a constant $C \geq \tilde{C} > 0$ only depending on n, p and $\|\phi\|_1$ such that*

$$\tilde{C}\sum_{j\geq\tilde{\ell}}\|\phi_j * u_0\|_p \leq \int_0^T \|\Delta e^{s\Delta}u_0\|_p ds \leq C\sum_{j\in\mathbb{Z}}\min(2^{2(j-\tilde{\ell})},1)\|\phi_j * u_0\|_p. \tag{2.6}$$

When we consider a time local problem to (1.3), then the initial data can be chosen in the inhomogeneous Besov space $B^0_{p,1}$. Indeed, we have the following:

Theorem 2.5. *Let $1 \leq p \leq \infty$ and for $T < \infty$ let $I = [0, T)$. For $u_0 \in B^0_{p,1}$, there exists $C_0 > 0$ and $C_T > 0$*

$$C_0\|u_0\|_{B^0_{p,1}} \leq \int_0^T \|\Delta e^{s\Delta}u_0\|_p ds \leq C_T\|u_0\|_{B^0_{p,1}},$$

where $C_0 \simeq C_T = O(\log T)$. In particular maximal L^1 regularity in the local interval holds for $I = [0, T)$. For the solution of the heat equation (1.3), there exists a constant $C_T > 0$ such that

$$\|\partial_t u\|_{L^1(I;B^0_{p,1})} + \|\nabla^2 u\|_{L^1(I;B^0_{p,1})} \leq C_T\Big(\|u_0\|_{B^0_{p,1}} + \|f\|_{L^1(I;B^0_{p,1})}\Big), \tag{2.7}$$

where $C_T = O(\log T)$ as $T \to \infty$. The estimate can be uniform in T if we exchange into the homogeneous Besov space $\dot{B}^0_{p,1}$.

Now we shall show the results for the Cauchy problem of the heat equation with constant coefficients in a slightly general setting. We consider the Cauchy problem of the parabolic equation with the fractional Laplacian $\mathcal{L}_\alpha = -(-\Delta)^{\alpha/2}$ with $\alpha > 0$:

$$\begin{cases} \partial_t u - \mathcal{L}_\alpha u = f, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.8}$$

Theorem 2.6 (optimal maximal L^1 regularity). *Let $\alpha > 0$ and $1 \leq p \leq \infty$. For $f \in L^1(\mathbb{R}_+; \dot{B}^0_{p,1}(\mathbb{R}^n))$ and $u_0 \in \dot{B}^0_{p,1}(\mathbb{R}^n)$ there exists a unique solution u to (2.8) which satisfies the estimate: There exists a positive constant $C_M > 0$ only depending on α, n, p such that*

$$\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})} + \|\mathcal{L}_\alpha u\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})} \leq C_M\left(\|u_0\|_{\dot{B}^0_{p,1}} + \|f\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})}\right). \tag{2.9}$$

Besides if $f \equiv 0$, then the regularity condition for the initial data is optimal. Namely there exists a constant $C_m = C_m(n,p) > 0$ such that for all $u_0 \in \dot{B}^0_{p,1}(\mathbb{R}^n)$

$$C_m\|u_0\|_{\dot{B}^0_{p,1}} \leq \|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})} + \|\mathcal{L}_\alpha u\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})}. \tag{2.10}$$

Theorem 2.1 is a direct consequence from Theorem 2.6 with $\alpha = 2$ and the boundedness of the singular integral operator from $\dot{B}^0_{p,1}$ to itself. This general form has some applications. See for instance Iwabuchi [18].

3. Results for a variable coefficient case

We consider the case where a coefficient is variable.

$$\begin{cases} \partial_t u - a(t,x)\Delta u = f, & t>0,\ x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n. \end{cases} \tag{3.1}$$

We assume that $a(t,x)$ satisfies the following:

(1) $a(t,x) = 1 + b(t,x)$,
(2) there exists $\underline{b} > -1$ s.t. $b(t,x) \ge \underline{b}$ a.e x,
(3) $b \in L^\infty(I; \dot{B}^{n/q}_{q,1}(\mathbb{R}^n)) \cap C(I; \dot{B}^{n/q}_{q,1}(\mathbb{R}^n))$ for $1 \le q < \infty$.

Theorem 3.1. *Let $1 \le p \le \infty$, $1 \le q < \infty$ and a variable coefficients $a(t,x)$ satisfies the assumption (1), (2), (3). For $T > 0$ we set $I = [0,T)$ and $\underline{\nu} := \inf_{t\in I, x\in\mathbb{R}^n}(1 + b(t,x))$. For $b \in L^\infty(I; \dot{B}^{n/q}_{q,1}(\mathbb{R}^n)) \cap C(I; \dot{B}^{n/q}_{q,1}(\mathbb{R}^n))$, $u_0 \in \dot{B}^0_{q,1}(\mathbb{R}^n)$ and $f \in L^1(0,T; \dot{B}^0_{p,1}(\mathbb{R}^n))$, there exists $C_M > 0$ the solution u to* (3.1) *satisfies the estimate:*

$$\begin{aligned} &\|\partial_t u\|_{L^1(0,T;\dot{B}^0_{p,1})} + \underline{\nu}\|\nabla^2 u\|_{L^1(0,T;\dot{B}^0_{p,1})} \\ &\le C_M \left\{1 + \|b\|_{L^\infty(I;\dot{B}^{n/q}_{q,1})} \exp\left(\mu T\big(1 + \|b\|_{L^\infty(I;\dot{B}^{n/q}_{q,1})}\big)^2\right)\right\} \|u_0\|_{\dot{B}^0_{p,1}} \\ &\quad + C_M \int_0^T \exp\left(\mu \int_s^T \big(1 + \|b(r)\|_{\dot{B}^{n/q}_{q,1}}\big)^2 dr\right) \|f(s)\|_{\dot{B}^0_{p,1}} ds, \end{aligned}$$

where $\mu = (CC_1\underline{\nu})^2 \log(1 + C_M)$.

Theorem 3.2. *Let $1 \le p \le \infty$, $1 \le q < \infty$ and a variable coefficients $a(t,x)$ satisfies the assumption (1), (2), (3). For $I = [0,T)$, we set $k = [-\frac{\log T}{2\log 2}]$. For $b \in L^\infty(I; \dot{B}^{n/q}_{q,1}(\mathbb{R}^n)) \cap C(I; \dot{B}^{n/q}_{q,1}(\mathbb{R}^n))$, $u_0 \in \dot{B}^0_{p,1}(\mathbb{R}^n)$,* (3.1) *with $f \equiv 0$ admits a unique solution u which satisfies*

$$\frac{C}{\big(1 + \|b\|_{L^\infty(I;\dot{B}^{n/q}_{q,1})}\big)} \sum_{\ell \ge k} \|\phi_\ell * u_0\|_p \le \Big(\|\partial_t u\|_{L^1(I;\dot{B}^0_{p,1})} + \|\nabla^2 u\|_{L^1(I;\dot{B}^0_{p,1})}\Big).$$

Theorem 3.2 shows that for $b \in L^\infty(I; \dot{B}^{n/q}_{q,1}(\mathbb{R}^n)) \cap C(I; \dot{B}^{n/q}_{q,1}(\mathbb{R}^n))$, the class $\dot{B}^0_{p,1}(\mathbb{R}^n)$ of u_0 could not be replaced by $L^p(\mathbb{R}^n)$, $\dot{B}^0_{p,\sigma}(\mathbb{R}^n)$, $\dot{F}^0_{p,\sigma}(\mathbb{R}^n)$ $(1 < \sigma \le \infty)$ for maximal L^1-regularity.

Danchin [9] and Haspot [17] obtained an analogous estimate for the variable coefficient case by an elegant usage of L^p type energy estimate and the Chemin-Laners spaces. In this case, the Chemin-Laners space coincides with the Bochner space as

$$\widetilde{L^1(I; \dot{B}^0_{p,1})} \equiv \ell^1(\{L^1(I; L^p_j)\}_{j\in\mathbb{Z}}) = L^1(I; \dot{B}^0_{p,1}),$$

thanks to the fact that the time L^1 norm and Littlewood-Paley sequence ℓ^1 norm can be interchanged, where L^p_j denotes the Littlewood-Paley decomposed L^p space given by $\|f\|_{L^p_j} \equiv \|\phi_j * f\|_p$. As in the constant coefficient case, our method is very much different from theirs. We use the estimate for the constant coefficient case (Theorem 2.1) and enploy a freezing arugment in space-time variables and then time variable to obtain the above result for variable coefficient. Our theorems Theorem 3.1 and 3.2 can be generalized for more general parabolic type equation with a second order uniformly elliptic operator $\mathcal{L}$:

(1) a parabolic system

$$\begin{cases} \partial_t u - \sum_{i,j=1}^{n} a_{ij}(t,x)\partial_i\partial_j u = f, & t>0,\ x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n, \end{cases}$$

where $a_{ij}(t,x)$ satisfies

(a) $a_{ij}(t,x) \in L^\infty(0,T;\dot{B}^{n/q}_{q,1}(\mathbb{R}^n)) \cap C(I;\dot{B}^{n/q}_{q,1}(\mathbb{R}^n))$, $\quad 1\le p,q\le\infty$,

(b) $a_{ij}(t,x) = \delta_{ij} + b_{ij}(t,x)$, $\quad 1\le i,j\le\infty$,

(c) $b_{ij}(t,x) = b_{ji}(t,x)$, $\quad 1\le i,j\le\infty$,

(d) there exists $\lambda\ge 0$ such that $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j \ge \lambda|\xi|^2$ for all $\xi\in\mathbb{R}^n$.

(2) the vector valued system such as the Stokes equation or the Lamé equation:

$$\begin{cases} \partial_t u - \Delta u + \nabla\pi = f, & t>0,\ x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n. \end{cases}$$

$$\begin{cases} \partial_t u - (\mu+\lambda)\Delta u + \lambda\nabla(\mathrm{div}\ u) = f, & t>0,\ x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n. \end{cases}$$

To treat the variable coefficients, we remark that the estimate in the Besov space such as

$$\|af\|_{\dot{B}^0_{p,1}} \le C\|a\|_\infty\|f\|_{\dot{B}^0_{p,1}}$$

fails in general. This is the reason why we adapt the space $\dot{B}^{n/q}_{q,1}(\mathbb{R}^n)$ for the variable coefficient which plays a role instead of L^∞ space.

Proposition 3.3. *Let $1\le p\le\infty$ and $1\le q<\infty$. For $f\in\dot{B}^{\frac{n}{q}}_{q,1}$ and $g\in\dot{B}^0_{p,1}$ there exists $C>0$ such that*

$$\|fg\|_{\dot{B}^0_{p,1}} \le C\|f\|_{\dot{B}^{\frac{n}{q}}_{q,1}}\|g\|_{\dot{B}^0_{p,1}}. \tag{3.2}$$

For the proof, we refer to Abidi-Paicu [1].

The space $\dot{B}^{n/q}_{q,1}(\mathbb{R}^n)$ has nice embedding property. Let

$$C_v(\mathbb{R}^n) = \{f\in C(\mathbb{R}^n) \mid |f(x)|\to 0 \text{ as } |x|\to\infty\}.$$

Proposition 3.4. *Let $1\le q<\infty$ and $\mathcal{S}(\mathbb{R}^n)$ be the rapidly decreasing smooth functions. Then*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \dot{B}^{n/q}_{q,1}(\mathbb{R}^n) \hookrightarrow C_v(\mathbb{R}^n). \tag{3.3}$$

In particular, the embedding of the left-hand side is dense.

Acknowledgments. The authors thank Professor Masashi Misawa for a helpful comment on the variable coefficient case. The work of the first author is partially supported by

JSPS, Grant-in-Aid for Scientific Research S #25220702. The work of the second author is partially suppored by JSPS, Grant-in-Aid for Scientific Reserch B #24340025.

References

[1] Abidi, H., Paicu, M., *Existence globale pour un fluide inhomogène*, Ann. Inst. Fourier (Grenoble), **57** (2007), 883–917,

[2] Amann, H., *Linear and Quasilinear Parabolic Problems. Vol I Abstract Linear Theory*, Monographs in Math. Vol **89**, Birkhäuser Verlag, Basel·Boston·Berlin, 1995.

[3] Bahouri, H., Chemin, J.-Y., Danchin, R., *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundelehren der Mathematischen Wissenschften, Springer 2011.

[4] Benedek, A., Calderón, A. P., Panzone, R., *Convolution operators on Banach space valued functions.* Proc. Nat. Acad. Sci. USA **48** (1962) 356–365.

[5] Bergh, J., Löfström, J., *Interpolation Spaces; an introduction*, Springer-Verlag, Berlin, 1976.

[6] Brenner, P., Thomée,. V. Wahlbin, L. *Besov spaces and applications to difference methods for initial value problems*, Lecture Notes in Math. **434** Springer-Verlag, 1975.

[7] Cannone, M., Planchon, F., Schonbek, M., *A critical functional framework for the inhomogeneous Navier-Stokes equations in the half-space*, Comm. in P.D.E., **25** (2000) 903-924.

[8] Clément, Ph., Prüss, J., *Global existence for a semilinear parabolic Volterra equation*, Math. Z., **209** (1992) 17-26.

[9] Danchin, R., *Density-dependent incompressible viscous fluids in critical spaces*, Proc. Roy Soc. Edinburgh **133A** (2003), 1311-1334.

[10] Danchin, R., *On the uniqueness in critical spaces for compressible Navier-Stokes equations*, Nonlinear Diff. Equations Appl., **12** (2005) 111-128.

[11] Danchin, R., *Well-posedness in critical spaces for barotropic viscous fluids with truly not constant density*, Comm. Partial Differential Equations, **32** (2007) 1373-1397.

[12] Danchin, R., Mucha, P. B., *A critical functional framework for the inhomogeneous Navier-Stokes equations in the half-space*, J. Funct. Anal., **256** (2009) 881-927.

[13] Da Prato, G., Grisvard, P., *Sommes d'opérateurs linéaires et équations différentielles opérationelles*, J. Math. Pure Appl. **54** (1975) 305-387.

[14] Denk, R., Hieber, M., Prüss, J., *$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Memoirs of AMS, **166**, No. 788 (2003).

[15] Dore, G., Venni, A., *On the closedness of the sum of two closed operators*, Math. Z., **196** (1987) 189-201.

[16] Giga, Y., Saal, J., *L^1 maximal regularity for the Lapalacian and applications*, Discrete Conti. Dyn. Syst. **I** (2011) 495-504.

[17] Haspot, B., *Well-posedness in critical spaces for the system of compressible Navier-Stokes in larger spaces*, J. Differential Equations, **251** (2011) 2262-2295.

[18] Iwabuchi, T., *Global solutions for the critical Burgers equation in the Besov spaces and the large time behavior*, Ann. I. H. Poincaré, (2014), to appear.

[19] Iwabuchi, T., *Maximal regularity for parabolic equation in the modulation spaces*, private communication.

[20] Kalton, N., Weis, L., *The H^∞-calculus and sums of closed operators*, Math. Ann., **321** (2001) 319–345.

[21] Ladyzhenskaya, O. A., Solonnikov, V. A., Ural'tseva, N. N., *Linear and quasilinear equations of parabolic type*, Amer. Math. Soc. Transl. Math. Monographs, Providence, R.I., 1968.

[22] Ogawa, T., Shimizu, S., *The drift diffusion system in two-dimensional critical Hardy space*, J. Funct. Anal., **255** (2008) 1107-1138.

[23] Ogawa, T., Shimizu, S., *End-point maximal regularity and its application to two-dimensional Keller-Segel system*, Math. Z., **264** (2010) 601-628.

[24] Ogawa, T., Shimizu, S., *End-point maximal L^1 regularity for a Cauchy problem to parabolic equations with variable coefficient*, Preprint.

[25] Triebel, H., *Theory of Function Spaces*, Birkhäuser, Basel, 1983.
[26] Weis, L., *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*, Math. Ann., **319** (2001) 735–758.